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Asymptotic analysis of solutions of systems of neutral functional differential equations

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Abstract

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Weakly coupled systems of neutral functional differential equations of the form

$$\frac{d^m}{dt^m}[x(t)-\lambda x(t-\sigma)]+f(t, x(\rho(t)), y(\theta(t)))=0,$$
$$\frac{d^n}{dt^n}[y(t)-\mu y(t-\tau)]+g(t, x(\rho(t)), y(\theta(t)))=0$$

are considered. We give explicit conditions under which the system has solutions with prescribed asymptotic behaviors.

Keywords: Delay differential equations; neutral delay equations; asymptotic behavior.

1. Introduction

Weakly coupled systems of neutral functional differential equations of the form

$$\begin{aligned}&\frac{d^m}{dt^m}[x(t)-\lambda x(t-\sigma)]+f(t, x(\rho(t)), y(\theta(t)))=0,\\&\frac{d^n}{dt^n}[y(t)-\mu y(t-\tau)]+g(t, x(\rho(t)), y(\theta(t)))=0\end{aligned}\tag{A}$$

will be considered under the following assumptions:

(a) m, n are positive integers and λ, μ, σ, τ are positive constants with $0<\lambda<1$ and $0<\mu<1$;

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(b) $\rho(t)$, $\theta(t)$ are continuous functions on $[t_0, \infty)$, $t_0 > 0$, such that $\lim_{t\to\infty} \rho(t) = \lim_{t\to\infty} \theta(t) = \infty$;

(c) $f(t, x, y)$, $g(t, x, y)$ are continuous functions on $[t_0, \infty) \times \mathbb{R}^2$ such that

$$|f(t, x, y)| \leqslant F(t, |x|, |y|), \qquad |g(t, x, y)| \leqslant G(t, |x|, |y|),$$

for $(t, x, y) \in [t_0, \infty) \times \mathbb{R}^2$ for some continuous functions $F(t, u, v)$, $G(t, u, v)$ on $[t_0, \infty) \times \mathbb{R}_+^2$ which are nondecreasing in u and v for each fixed $t \geqslant t_0$.

Our main interest is to study the existence and asymptotic behavior of solutions $(x(t), y(t))$ of (A) defined in some neighborhood of infinity. Since the unperturbed equations

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}[x(t) - \lambda x(t-\sigma)] = 0 \quad \text{and} \quad \frac{\mathrm{d}^n}{\mathrm{d}t^n}[y(t) - \mu y(t-\tau)] = 0$$

have the solutions

$$\{1, t, \ldots, t^{m-1}, \lambda^{t/\sigma}\omega_\sigma(t)\} \quad \text{and} \quad \{1, t, \ldots, t^{n-1}, \mu^{t/\tau}\omega_\tau(t)\},$$

$\omega_\sigma(t)$ [respectively $\omega_\tau(t)$] being any continuous σ-periodic [respectively τ-periodic] function, it will be natural to expect that there exist solutions $(x(t), y(t))$ of the system (A) having the following three types of asymptotic behavior as $t \to \infty$:

(I) $x(t) = \text{const.} \cdot t^j + o(t^j)$ and $y(t) = \text{const.} \cdot t^k + o(t^k)$, as $t \to \infty$, for any given $j \in \{0, 1, \ldots, m-1\}$ and $k \in \{0, 1, \ldots, n-1\}$;

(II) $x(t) = \text{const.} \cdot \lambda^{t/\sigma}\omega_\sigma(t) + o(\lambda^{t/\sigma})$ and $y(t) = \text{const.} \cdot \mu^{t/\tau}\omega_\tau(t) + o(\mu^{t/\tau})$, as $t \to \infty$, for any given continuous σ-periodic function $\omega_\sigma(t)$ and any τ-periodic function $\omega_\tau(t)$;

(III) $x(t) = \text{const.} \cdot t^j + o(t^j)$ and $y(t) = \text{const.} \cdot \mu^{t/\tau}\omega_\tau(t) + o(\mu^{t/\tau})$, as $t \to \infty$, for any given $j \in \{0, 1, \ldots, m-1\}$ and any continuous τ-periodic function $\omega_\tau(t)$.

An analysis will be given to yield explicit conditions under which (A) has solutions with the asymptotic behavior (I), (II) or (III) described above. Every solution of type (I) is nonoscillatory in the sense that both of its components are nonoscillatory. A solution of type (II) is either nonoscillatory or oscillatory according to whether the involved periodic functions $\omega_\sigma(t)$ and $\omega_\tau(t)$ are both nonoscillatory or oscillatory in the usual sense. Since the existence criterion for type (II) solutions of (A) is independent of $\omega_\sigma(t)$ and $\omega_\tau(t)$, it turns out that the system (A) may possess both oscillatory and nonoscillatory solutions. In addition there is a situation in which (A) has all the three types of solutions simultaneously, implying automatically the coexistence of oscillatory solutions and nonoscillatory solutions.

It seems to us that very little is known about the oscillatory and nonoscillatory behavior of systems of neutral functional differential equations, though single neutral equations have been intensively studied in recent years (see, e.g., [1–4, 6–8]). The only reference we are aware of is [5] in which bounded nonoscillatory solutions are constructed for a class of neutral systems similar to (A).

2. Existence of solutions of type (I)

2.1. Main result

The first result of this paper concerns type (I) solutions of (A).

Theorem 1. *Let $j \in \{0, 1, \ldots, m-1\}$, $k \in \{0, 1, \ldots, n-1\}$ and suppose that there is a constant $a > 0$ such that*

$$\int_{t_0}^{\infty} t^{m-j-1} F\big(t, a[\rho(t)]^j, a[\theta(t)]^k\big)\,\mathrm{d}t < \infty,$$
$$\int_{t_0}^{\infty} t^{n-k-1} G\big(t, a[\rho(t)]^j, a[\theta(t)]^k\big)\,\mathrm{d}t < \infty. \tag{2.1}$$

Then the system (A) *has a solution* $(x(t), y(t))$ *with the property*

$$x(t) = \alpha t^j + \mathrm{o}(t^j) \quad \text{and} \quad y(t) = \beta t^k + \mathrm{o}(t^k), \qquad \text{as } t \to \infty, \tag{2.2}$$

for some nonzero constants α and β.

2.2. Lemmas

We first state two lemmas which will be needed in proving the above theorem.

Let $C[T, \infty)$ denote the set of all continuous functions $\xi : [T, \infty) \to \mathbb{R}$ and let $C_{\lambda,\sigma}[T, \infty)$ be the subset of $C[T, \infty)$ consisting of the functions η such that $\sum_{i=1}^{\infty} \lambda^{-i}\eta(t + i\sigma)$ is uniformly convergent on compact subintervals of $[T - \sigma, \infty)$. Define the mappings $\Phi_{\lambda,\sigma} : C[T, \infty) \to C[T - \sigma, \infty)$ and $\Psi_{\lambda,\sigma} : C_{\lambda,\sigma}[T, \infty) \to C[T - \sigma, \infty)$ by the following formulas:

$$\Phi_{\lambda,\sigma}\xi(t) = \sum_{i=0}^{n(t)-1} \lambda^i \xi(t - i\sigma) + \frac{\lambda^{n(t)}\xi(T)}{1-\lambda}, \quad t > T,$$
$$\Phi_{\lambda,\sigma}\xi(t) = \frac{\xi(T)}{1-\lambda}, \qquad T - \sigma \leqslant t \leqslant T, \tag{2.3}$$

$n(t)$ being the smallest positive integer such that $t - n(t)\sigma \leqslant T$, and

$$\Psi_{\lambda,\sigma}\eta(t) = \sum_{i=1}^{\infty} \lambda^{-i}\eta(t + i\sigma), \quad t \geqslant T - \sigma. \tag{2.4}$$

Lemma 2. (i) *If $\xi \in C[T, \infty)$, then $x = \Phi_{\lambda,\sigma}\xi$ satisfies*

$$x(t) - \lambda x(t - \sigma) = \xi(t), \quad t \geqslant T.$$

(ii) *If $\eta \in C_{\lambda,\sigma}[T, \infty)$, then $y = \Psi_{\lambda,\sigma}\eta$ satisfies*

$$y(t) - \lambda y(t - \sigma) = -\eta(t), \quad t \geqslant T.$$

Lemma 3. *Let $j \geqslant 0$ be an integer. Then, for $x \in C[T, \infty)$,*

$$\lim_{t\to\infty} \frac{x(t) - \lambda x(t-\sigma)}{t^j} = c \neq 0 \quad \textit{implies} \quad \lim_{t\to\infty} \frac{x(t)}{t^j} = \frac{c}{1-\lambda}.$$

Lemma 2 is almost trivial. The proof of Lemma 3 can be found in [5].

Proof of Theorem 1. The proof of Theorem 1 is based on a new idea of applying the well-known Schauder–Tychonoff fixed-point theorem.

Let $c>0$ be a constant such that $2c/(1-\lambda)\leqslant a$, $2c/(1-\mu)\leqslant a$ and choose $T>t_0$ large enough such that

$$T_* = \min\left\{T-\sigma,\ T-\tau,\ \inf_{t\geqslant T}\rho(t),\ \inf_{t\geqslant T}\theta(t)\right\}\geqslant t_0 \tag{2.5}$$

and

$$\begin{aligned} &\int_T^\infty t^{m-j-1}F\left(t,\ a[\sigma(t)]^j,\ a[\theta(t)]^k\right)\mathrm{d}t\leqslant \tfrac{1}{2}c,\\ &\int_T^\infty t^{n-k-1}G\left(t,\ a[\sigma(t)]^j,\ a[\theta(t)]^k\right)\mathrm{d}t\leqslant \tfrac{1}{2}c. \end{aligned} \tag{2.6}$$

Let X, Y, U, V be the set of continuous functions defined by

$$\begin{aligned} X&=\left\{x\in C[T_*,\infty): \frac{c}{2j!}(t-T)_+^j\leqslant x(t)\leqslant\frac{2c}{(1-\lambda)j!}(t-T)_+^j,\ t\geqslant T_*\right\},\\ Y&=\left\{y\in C[T_*,\infty): \frac{c}{2k!}(t-T)_+^k\leqslant y(t)\leqslant\frac{2c}{(1-\mu)k!}(t-T)_+^k,\ t\geqslant T_*\right\}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} U&=\left\{u\in C[T,\infty): \frac{c}{2j!}(t-T)^j\leqslant u(t)\leqslant\frac{2c}{j!}(t-T)^j,\ t\geqslant T\right\},\\ V&=\left\{v\in C[T,\infty): \frac{c}{2k!}(t-T)^k\leqslant v(t)\leqslant\frac{2c}{k!}(t-T)^k,\ t\geqslant T\right\}, \end{aligned} \tag{2.8}$$

where $(t-T)_+=t-T$ if $t\geqslant T$ and $(t-T)_+=0$ if $t<T$, and it is understood that $(t-T)_+^0=1$. We use the notation $Z=X\times Y$, $W=U\times V$. It is clear that Z and W are closed convex subsets of the Fréchet spaces $C[T_*,\infty)^2$, and $C[T,\infty)^2$, respectively.

Let us define $\mathscr{F}_1: W\to C[T_*,\infty)^2$ to be a mapping which assigns to each $w=(u,v)\in W$ a vector function $z=(x,y)$, where

$$\begin{aligned} &x(t)=\Phi_{\lambda,\sigma}u(t),\quad t\geqslant T-\sigma,\quad x(t)=\Phi_{\lambda,\sigma}u(T-\sigma),\quad T_*\leqslant t\leqslant T-\sigma,\\ &y(t)=\Phi_{\mu,\tau}v(t),\quad t\geqslant T-\tau,\quad y(t)=\Phi_{\mu,\tau}v(T-\tau),\quad T_*\leqslant t\leqslant T-\tau, \end{aligned} \tag{2.9}$$

and denote by $\mathscr{F}_2: Z\to C[T,\infty)^2$ a mapping which assigns to each $z=(x,y)\in Z$ a vector function $w=(u,v)$ given by

$$\begin{aligned} u(t)&=\frac{c}{j!}(t-T)^j+I_{m,j}(f;x,y)(t),\quad t\geqslant T,\\ v(t)&=\frac{c}{k!}(t-T)^k+I_{n,k}(g;x,y)(t),\quad t\geqslant T, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} &I_{m,0}(f;x,y)(t)=(-1)^{m-1}\int_t^\infty\frac{(s-t)^{m-1}}{(m-1)!}f(s,x(\rho(s)),y(\theta(s)))\,\mathrm{d}s,\\ &I_{m,j}(f;x,y)(t)\\ &\quad=(-1)^{m-j-1}\int_T^t\frac{(t-s)^{j-1}}{(j-1)!}\int_s^\infty\frac{(r-s)^{m-j-1}}{(m-j-1)!}f(r,x(\rho(r)),y(\theta(r)))\,\mathrm{d}r\,\mathrm{d}s,\\ &\quad j\in\{1,\ldots,m-1\}, \end{aligned} \tag{2.11}$$

and $I_{n,k}(g;\, x,\, y)$ are defined similarly. Now we define $\mathscr{F}: \boldsymbol{Z}\times \boldsymbol{W} \to C[T_*, \infty)^2 \times C[T, \infty)^2$ by

$$\mathscr{F}(z, w) = (\mathscr{F}_1 w,\, \mathscr{F}_2 z), \quad (z, w) \in \boldsymbol{Z}\times \boldsymbol{W}. \tag{2.12}$$

It is a matter of simple computation to show that $\mathscr{F}_1(\boldsymbol{W}) \subset \boldsymbol{Z}$ and $\mathscr{F}_2(\boldsymbol{Z}) \subset \boldsymbol{W}$, so that $\mathscr{F}$ maps $\boldsymbol{Z}\times \boldsymbol{W}$ into itself. It can also be shown without difficulty that $\mathscr{F}$ is continuous and $\mathscr{F}(\boldsymbol{Z}\times \boldsymbol{W})$ is relatively compact in the topology of $C[T_*, \infty)^2 \times C[T, \infty)^2$. Therefore, by the Schauder–Tychonoff fixed-point theorem, there exists an element $(z, w) = (x, y;\, u, v) \in \boldsymbol{Z}\times \boldsymbol{W}$ such that $(z, w) = \mathscr{F}(z, w)$, that is, $z = \mathscr{F}_1 w$ and $w = \mathscr{F}_2 z$ because of (2.12). This then implies that

$$x(t) = \Phi_{\lambda,\sigma} u(t), \qquad y(t) = \Phi_{\mu,\tau} v(t), \tag{2.13}$$

$$u(t) = \frac{c(t-T)^j}{j!} + I_{m,j}(f;\, x,\, y)(t), \qquad v(t) = \frac{c(t-T)^k}{k!} + I_{n,k}(g;\, x,\, y)(t), \tag{2.14}$$

for $t \geqslant T$, from which in view of Lemma 2 (i) and the definition of $I_{m,j}(f;\, x,\, y)$, $I_{n,k}(g;\, x,\, y)$ it follows that

$$x(t) - \lambda x(t-\sigma) = u(t), \qquad y(t) - \mu y(t-\tau) = v(t), \tag{2.15}$$

$$u^{(m)}(t) = -f(t,\, x(\rho(t)),\, y(\theta(t))), \qquad v^{(n)}(t) = -g(t,\, x(\rho(t)),\, y(\theta(t))), \tag{2.16}$$

for $t \geqslant T$. From (2.15) and (2.16) we see that $(x(t), y(t))$ is a solution of the system (A) for $t \geqslant T$. Noting that $\lim_{t\to\infty} u(t)/t^j = c/j!$, $\lim_{t\to\infty} v(t)/t^k = c/k!$ by (2.14) and (2.11), and applying Lemma 3 to (2.15), we conclude that

$$\lim_{t\to\infty} \frac{x(t)}{t^j} = \frac{c}{(1-\lambda)j!} \quad \text{and} \quad \lim_{t\to\infty} \frac{y(t)}{t^k} = \frac{c}{(1-\mu)k!},$$

showing that the solution $(x(t), y(t))$ of (A) has the desired asymptotic property (2.2). This completes the proof. □

3. Existence of solutions of types (II) and (III)

3.1. Main results

The main results of this section are as follows.

Theorem 4. *Suppose that there is a constant $a > 0$ such that*

$$\begin{aligned} &\int_{t_0}^{\infty} \lambda^{-t/\sigma} F(t,\, a\lambda^{\rho(t)/\sigma},\, a\mu^{\theta(t)/\tau})\, \mathrm{d}t < \infty, \\ &\int_{t_0}^{\infty} \mu^{-t/\tau} G(t,\, a\lambda^{\rho(t)/\sigma},\, a\mu^{\theta(t)/\tau})\, \mathrm{d}t < \infty. \end{aligned} \tag{3.1}$$

Then, given a continuous σ-periodic function $\omega_\sigma(t)$ and a continuous τ-periodic function $\omega_\tau(t)$, the system (A) *possesses a solution $(x(t), y(t))$ with the property*

$$x(t) = \alpha\lambda^{t/\sigma}\omega_\sigma(t) + o(\lambda^{t/\sigma}), \quad y(t) = \beta\mu^{t/\tau}\omega_\tau(t) + o(\mu^{t/\tau}), \qquad \text{as } t \to \infty, \tag{3.2}$$

for some nonzero constants α and β.

Theorem 5. *Let $j \in \{0, 1, \ldots, m-1\}$ and suppose that there is a constant $a > 0$ such that*

$$\begin{aligned} &\int_{t_0}^{\infty} t^{m-j-1}F\big(t, a[\rho(t)]^j, a\mu^{\theta(t)/\tau}\big)\,\mathrm{d}t < \infty, \\ &\int_{t_0}^{\infty} \mu^{-t/\tau}G\big(t, a[\rho(t)]^j, a\mu^{\theta(t)/\tau}\big)\,\mathrm{d}t < \infty. \end{aligned} \tag{3.3}$$

Then for a given continuous τ-periodic function $\omega_\tau(t)$, the system (A) *possesses a solution $(x(t), y(t))$ with the property*

$$x(t) = \alpha t^j + o(t^j), \quad y(t) = \beta\mu^{t/\tau}\omega_\tau(t) + o(\mu^{t/\tau}), \qquad \text{as } t \to \infty, \tag{3.4}$$

for some constants α and β.

3.2. Lemma

We begin by stating a result which will play a crucial role in the proof of the above theorems.

Lemma 6. *Let $h(t) \geq 0$ be a continuous function on $[T, \infty)$ such that $\int_T^\infty \lambda^{-t/\sigma}h(t)\,\mathrm{d}t < \infty$. Then, the function $H(t) = \int_t^\infty (s-t)^{m-1}h(s)\,\mathrm{d}s$ satisfies $\Psi_{\lambda,\sigma}H(t) = o(\lambda^{t/\sigma})$, as $t \to \infty$, where $\Psi_{\lambda,\sigma}$ is defined by* (2.4).

Proof. First note that since $\lambda < 1$, $H(t)$ is well defined for $t \geq T$. We have

$$\begin{aligned} 0 \leq \Psi_{\lambda,\sigma}H(t) &= \sum_{i=1}^{\infty} \lambda^{-i}\int_{t+i\sigma}^{\infty} (s-t-i\sigma)^{m-1}h(s)\,\mathrm{d}s \\ &= \sum_{i=1}^{\infty} \lambda^{-i}\sum_{j=i}^{\infty}\int_{t+j\sigma}^{t+(j+1)\sigma} (s-t-i\sigma)^{m-1}h(s)\,\mathrm{d}s \\ &= \sum_{j=1}^{\infty}\int_{t+j\sigma}^{t+(j+1)\sigma}\sum_{i=1}^{j} (s-t-i\sigma)^{m-1}\lambda^{-i}h(s)\,\mathrm{d}s \\ &= \sum_{j=1}^{\infty}\int_{t+j\sigma}^{t+(j+1)\sigma}\sum_{i=1}^{j} (s-t-i\sigma)^{m-1}\lambda^{j-i}\lambda^{-j}h(s)\,\mathrm{d}s. \end{aligned}$$

It can be shown that there is a constant $M > 0$ (independent of j) such that for each j,

$$s \in [t+j\sigma, t+(j+1)\sigma] \quad \text{implies} \quad \sum_{i=1}^{j} (s-t-i\sigma)^{m-1}\lambda^{j-i} \leq M.$$

In fact, if $t+j\sigma \leqslant s \leqslant t+(j+1)\sigma$, then

$$\begin{aligned}\sum_{i=1}^{j}(s-t-i\sigma)^{m-1}\lambda^{j-1} &= \sum_{i=1}^{j}\left[(s-t-j\sigma)+(j-i)\sigma\right]^{m-1}\lambda^{j-i}\\ &\leqslant \sum_{i=1}^{j}\left[(2(s-t-i\sigma))^{m-1}+(2(j-i)\sigma)^{m-1}\right]\lambda^{j-i}\\ &\leqslant (2\sigma)^{m-1}\sum_{i=1}^{j}\left[1+(j-i)^{m-1}\right]\lambda^{j-i}\\ &\leqslant (2\sigma)^{m-1}\sum_{k=0}^{\infty}(1+k^{m-1})\lambda^{k}\equiv M.\end{aligned}$$

It follows that

$$\begin{aligned}0 &\leqslant \Psi_{\lambda,\sigma}H(t) \leqslant M\sum_{j=1}^{\infty}\int_{t+j\sigma}^{t+(j+1)\sigma}\lambda^{-j}h(s)\,\mathrm{d}s\\ &\leqslant M\sum_{j=1}^{\infty}\int_{t+j\sigma}^{t+(j+1)\sigma}\lambda^{-(s-t)/\sigma}h(s)\,\mathrm{d}s = M\lambda^{t/\sigma}\int_{t+\sigma}^{\infty}\lambda^{-s/\sigma}h(s)\,\mathrm{d}s,\end{aligned}$$

proving Lemma 6. □

Proof of Theorem 4. For simplicity we put

$$F(t)=F\left(t,\ a\lambda^{\rho(t)/\sigma},\ a\mu^{\theta(t)/\tau}\right),\qquad G(t)=G\left(t,\ a\lambda^{\rho(t)/\sigma},\ a\mu^{\theta(t)/\tau}\right).$$

Let $c>0$ be so small that $c(1+\max|\omega_\sigma(t)|)\leqslant a$ and $c(1+\max|\omega_\tau(t)|)\leqslant a$, and choose T so large that (2.5) holds and

$$\begin{aligned}&\Psi_{\lambda,\sigma}\left[\int_t^{\infty}(s-t)^{m-1}F(s)\,\mathrm{d}s\right]\leqslant c\lambda^{t/\sigma},\\ &\Psi_{\mu,\tau}\left[\int_t^{\infty}(s-t)^{n-1}G(s)\,\mathrm{d}s\right]\leqslant c\mu^{t/\tau},\end{aligned}\tag{3.5}$$

for $t\geqslant T_*$. Formula (3.5) is a consequence of Lemma 6. Define

$$\begin{aligned}&X=\left\{x\in C[T_*,\infty):|x(t)|\leqslant a\lambda^{t/\sigma},\ t\geqslant T_*\right\},\\ &Y=\left\{y\in C[T_*,\infty):|y(t)|\leqslant a\mu^{t/\tau},\ t\geqslant T_*\right\},\end{aligned}\tag{3.6}$$

$$\begin{aligned}&U=\left\{u\in C[T,\infty):|u(t)|\leqslant\int_t^{\infty}(s-t)^{m-1}F(s)\,\mathrm{d}s,\ t\geqslant T\right\},\\ &V=\left\{v\in C[T,\infty):|v(t)|\leqslant\int_t^{\infty}(s-t)^{n-1}G(s)\,\mathrm{d}s,\ t\geqslant T\right\}.\end{aligned}\tag{3.7}$$

Let $\mathscr{F}_1: W=U\times V\to C[T_*,\infty)^2$ be a mapping which assigns to each $w=(u,v)\in W$ a vector function $z=(x,y)$, where

$$\begin{aligned}&x(t)=\begin{cases}c\lambda^{t/\sigma}\omega_\sigma(t)-\Psi_{\lambda,\sigma}u(t), & t\geqslant T-\sigma,\\ c\lambda^{t/\sigma}\omega_\sigma(t)-\Psi_{\lambda,\sigma}u(T-\sigma), & T_*\leqslant t\leqslant T-\sigma,\end{cases}\\ &y(t)=\begin{cases}c\mu^{t/\tau}\omega_\tau(t)-\Psi_{\mu,\tau}v(t), & t\geqslant T-\tau,\\ c\mu^{t/\tau}\omega_\tau(t)-\Psi_{\mu,\tau}v(T-\tau), & T_*\leqslant t\leqslant T-\tau,\end{cases}\end{aligned}\tag{3.8}$$

and let $\mathscr{F}_2 : Z = X \times Y \to C[T, \infty)^2$ be a mapping which assigns to each $z = (x, y) \in Z$ a vector function $w = (u, v)$, where

$$u(t) = I_{m,0}(f; x, y)(t), \quad v(t) = I_{n,0}(g; x, y)(t), \qquad t \geq T. \tag{3.9}$$

(See (2.11) for the definition of $I_{m,0}(f; x, y)$ and $I_{n,0}(g; x, y)$.) Consider the mapping $\mathscr{F} : Z \times W \to C[T_*, \infty)^2 \times C[T, \infty)^2$ defined by (2.12). Then, $\mathscr{F}$ maps $Z \times W$ into itself, since $\mathscr{F}_2(Z) \subset W$ by (3.6) and (3.9), and $\mathscr{F}_1(W) \subset Z$ by (3.7), (3.8) and (3.5). Since the continuity of $\mathscr{F}$ and the relative compactness of $\mathscr{F}(Z \times W)$ can be verified in a routine manner, $\mathscr{F}$ has a fixed element $(z, w) = (x, y; u, v) \in Z \times W$ which clearly satisfies

$$x(t) = c\lambda^{t/\sigma}\omega_\sigma(t) - \Psi_{\lambda,\sigma}u(t), \quad y(t) = c\mu^{t/\tau}\omega_\tau(t) - \Psi_{\mu,\tau}v(t), \quad t \geq T,$$
$$u(t) = I_{m,0}(f; x, y)(t), \qquad v(t) = I_{n,0}(g; x, y)(t), \qquad t \geq T.$$

It follows that $(x(t), y(t))$ is a solution of the system (A) for $t \geq T$. To see that $(x(t), y(t))$ satisfies (3.2) it suffices to notice that $\Psi_{\lambda,\sigma}u(t) = o(t^{t/\sigma})$ and $\Psi_{\mu,\tau}v(t) = o(\mu^{t/\tau})$, as $t \to \infty$, on account of Lemma 6. This completes the proof. □

Proof of Theorem 5. Put

$$F(t) = F\left(t, a[\rho(t)]^j, a\mu^{\theta(t)/\tau}\right), \qquad G(t) = G\left(t, a[\rho(t)]^j, a\mu^{\theta(t)/\tau}\right).$$

Let $c > 0$ be a constant such that $2c/(1-\lambda) \leq a$ and $c(1 + \max|\omega_\tau(t)|) \leq a$, and choose T so large that (2.5) holds and

$$\int_T^\infty t^{m-j-1}F(t)\,dt \leq \tfrac{1}{2}c \quad \text{and} \quad \Psi_{\mu,\tau}\left[\int_t^\infty (s-t)^{n-1}G(s)\,ds\right] \leq c\mu^{t/\tau}, \qquad t \geq T_*.$$

Consider the sets $Z = X \times Y$ and $W = U \times V$, where

$$X = \left\{x \in C[T_*, \infty) : \frac{c}{2j!}(t-T)_+^j \leq x(t) \leq \frac{2c}{(1-\lambda)j!}(t-T)_+^j,\ t \geq T_*\right\},$$
$$Y = \left\{y \in C[T_*, \infty) : |y(t)| \leq a\mu^{t/\tau},\ t \geq T_*\right\},$$
$$U = \left\{u \in C[T, \infty) : \frac{c}{2j!}(t-T)^j \leq u(t) \leq \frac{2c}{j!}(t-T)^j,\ t \geq T\right\},$$
$$V = \left\{v \in C[T, \infty) : |v(t)| \leq \int_t^\infty (s-t)^{n-1}G(s)\,ds,\ t \geq T\right\},$$

and let $\mathscr{F}_1 : W \to C[T_*, \infty)^2$ and $\mathscr{F}_2 : Z \to C[T, \infty)^2$ denote the mappings defined as follows: $\mathscr{F}_1 w = \mathscr{F}_1(u, v) = (x, y)$, where

$$x(t) = \begin{cases} \Phi_{\lambda,\sigma}u(t), & t \geq T - \sigma, \\ \Phi_{\lambda,\sigma}u(T-\sigma), & T_* \leq t \leq T - \sigma, \end{cases}$$

$$y(t) = \begin{cases} c\mu^{t/\tau}\omega_\tau(t) - \Psi_{\mu,\tau}v(t), & t \geq T - \tau, \\ c\mu^{t/\tau}\omega_\tau(t) - \Psi_{\mu,\tau}v(T-\tau), & T_* \leq t \leq T - \tau. \end{cases}$$

$\mathscr{F}_2 z = \mathscr{F}_2(x, y) = (u, v)$, where

$$u(t) = \frac{c(t-T)^j}{j!} + I_{m,j}(f; x, y)(t), \quad t \geq T,$$
$$v(t) = I_{n,0}(g; x, y)(t), \qquad t \geq T.$$

$\mathscr{F}_2 z = \mathscr{F}_2(x, y) = (u, v)$, where

$$u(t) = \frac{c(t-T)^j}{J!} + I_{m,j}(f;\, x, y)(t), \quad t \geqslant T,$$
$$v(t) = I_{n,0}(g;\, x, y)(t), \quad t \geqslant T.$$

Then proceeding as in the proofs of Theorems 1 and 4, we can show that there exists a $(z, w) = (x, y;\, u, v) \in Z \times W$ such that $\mathscr{F}_1(u, v) = (x, y)$ and $\mathscr{F}_2(x, y) = (u, v)$, and consequently $z = (x, y)$ gives a solution of the system (A) for $t \geqslant T$ having the asymptotic property (3.5). This sketches the proof of Theorem 5. □

Remark 7. We note the component $x(t)$ [respectively $y(t)$] of the solution $(x(t), y(t))$ of (A) established in Theorem 4 is oscillatory or nonoscillatory according to whether the periodic function $\omega_\sigma(t)$ [respectively $\omega_\tau(t)$] is oscillatory or nonoscillatory. Since $\omega_\sigma(t)$ and $\omega_\tau(t)$ can be given arbitrarily and since they do not appear explicity in (3.1), under the condition (3.1) the system (A) possesses the following three types of solutions:

(i) the solutions, both components of which are oscillatory;
(ii) the solutions, exactly one component of which is oscillatory;
(iii) the solutions, both components of which are nonoscillatory.

Since λ and μ are assumed to be less than 1, in any of the above cases, both components of the solutions decay to zero as $t \to \infty$.

We conclude with an example illustrating the results developed above.

Example 8. Consider the system of neutral equations

$$\begin{aligned} &\frac{d^n}{dt^n}[x(t) - \lambda x(t-1)] + a(t)x^\alpha(t-2) + b(t)y^\beta(t-3) = 0, \\ &\frac{d^n}{dt^n}[y(t) - \mu y(t-1)] + c(t)x^\gamma(t-2) + d(t)y^\delta(t-3) = 0, \end{aligned} \tag{3.10}$$

where $0 < \lambda < 1$, $0 < \mu < 1$, α, β, γ, δ are ratios of odd positive integers, and $a(t)$, $b(t)$, $c(t)$, $d(t)$ are continuous functions on $[t_0, \infty)$. Suppose that

$$\int_{t_0}^\infty t^{m-j-1+\alpha j}\,|a(t)|\,dt < \infty, \qquad \int_{t_0}^\infty t^{m-j-1+\beta k}\,|b(t)|\,dt < \infty,$$
$$\int_{t_0}^\infty t^{n-k-1+\gamma j}\,|c(t)|\,dt < \infty, \qquad \int_{t_0}^\infty t^{n-k-1+\delta k}\,|d(t)|\,dt < \infty,$$

for some $j \in \{0, 1, \ldots, m-1\}$ and $k \in \{0, 1, \ldots, n-1\}$. Then, Theorem 1 implies that system (3.10) has a solution $(x(t), y(t))$ such that

$$x(t) = \text{const.} \cdot t^j + o(t^j) \quad \text{and} \quad y(t) = \text{const.} \cdot t^k + o(t^k), \qquad \text{as } t \to \infty. \tag{3.11}$$

If, in particular, (3.10) is linear (i.e., $\alpha=\beta=\gamma=\delta=1$) and

$$\begin{aligned} &\int_{t_0}^{\infty} t^{m-1}|a(t)|\,dt<\infty, && \int_{t_0}^{\infty} t^{m+n-2}|b(t)|\,dt<\infty,\\ &\int_{t_0}^{\infty} t^{m+n-2}|c(t)|\,dt<\infty, && \int_{t_0}^{\infty} t^{n-1}|d(t)|\,dt<\infty, \end{aligned} \tag{3.12}$$

then, for any $j\in\{0,1,\ldots,m-1\}$ and $k\in\{0,1,\ldots,n-1\}$, there exists a solution $(x(t), y(t))$ of (3.10) satisfying (3.11).

Next suppose that

$$\begin{aligned} &\int_{t_0}^{\infty} \lambda^{(\alpha-1)t}|a(t)|\,dt<\infty, && \int_{t_0}^{\infty} \left(\frac{\mu^{\beta}}{\lambda}\right)^{t}|b(t)|\,dt<\infty,\\ &\int_{t_0}^{\infty} \left(\frac{\lambda^{\gamma}}{\mu}\right)^{t}|c(t)|\,dt<\infty, && \int_{t_0}^{\infty} \mu^{(\delta-1)t}|d(t)|\,dt<\infty. \end{aligned} \tag{3.13}$$

Then, according to Theorem 4, system (3.10) has a solution $(x(t), y(t))$ such that

$$x(t)=\text{const.}\cdot\lambda^{t}\omega_1(t)+o(\lambda^{t}) \quad\text{and}\quad y(t)=\text{const.}\cdot\mu^{t}\omega_2(t)+o(\mu^{t}), \tag{3.14}$$

as $t\to\infty$, for any given periodic functions $\omega_1(t)$ and $\omega_2(t)$ with period 1. The component $x(t)$ [respectively $y(t)$] is oscillatory or nonoscillatory according to whether $\omega_1(t)$ [respectively $\omega_2(t)$] is oscillatory or nonoscillatory. Typical examples of oscillatory 1-periodic solution are $\cos 2k\pi t$ and $\sin 2k\pi t$, $k=1,2,\ldots$. If, in particular, (3.9) is linear and $\lambda=\mu$, then (3.13) reduces to

$$\int_{t_0}^{\infty}(|a(t)|+|b(t)|+|c(t)|+|d(t)|)\,dt<\infty, \tag{3.15}$$

which ensures the existence of a solution $(x(t), y(t))$ satisfying (3.14) with $\lambda=\mu$. Since (3.12) implies (3.15), under (3.12) the linear system (3.10) (with $\lambda=\mu$) possesses nonoscillatory solutions satisfying (3.11) for any $j\in\{0,1,\ldots,m-1\}$ and $k\in\{0,1,\ldots,n-1\}$ as well as solutions, both oscillatory and nonoscillatory, which decay exponentially to zero as $t\to\infty$.

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